Analysis of Temporal Interactions with Link Streams

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interactions over time

- a, b, c, and d for 10 time units
interactions over time

- $a$, $b$, $c$, and $d$ for 10 time units
- $a$ always present, $b$ leaves from 4 to 5, $c$ present from 4 to 9, $d$ from 1 to 3
interactions over time

- $a$, $b$, $c$, and $d$ for 10 time units
- $a$ always present, $b$ leaves from 4 to 5, $c$ present from 4 to 9, $d$ from 1 to 3
- $a$ and $b$ interact from 1 to 3 and from 7 to 8; $b$ and $c$ from 6 to 9; $b$ and $d$ from 2 to 3.
interactions over time

- a, b, c, and d for 10 time units
- a always present, b leaves from 4 to 5, c present from 4 to 9, d from 1 to 3
- a and b interact from 1 to 3 and from 7 to 8; b and c from 6 to 9; b and d from 2 to 3.

e.g., social interactions, network traffic, money transfers, chemical reactions, etc.
• $a$, $b$, $c$, and $d$ for 10 time units
• $a$ always present, $b$ leaves from 4 to 5, $c$ present from 4 to 9, $d$ from 1 to 3
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e.g., social interactions, network traffic, money transfers, chemical reactions, etc.

how to describe such data?
structure or dynamics

信号分析，时间序列 → 动态

图论
网络科学

- 结构
- 动态

- 图序列
- 时间片
- 信息损失
- 什么片？

度数
密度
路径
structure and dynamics?

Time slices → graph sequence
structure and dynamics?

signal analysis, time series \(\rightarrow\) dynamics

time slices \(\rightarrow\) graph sequence

information loss
what slices?
graph sequences?
structure and dynamics

MAG / temporal graphs

TVG

lossless but graph-oriented

+ ad-hoc properties (mostly path-related) + contact sequences + relational event models + ...
structure and dynamics

MAG / temporal graphs

lossless but graph-oriented

+ ad-hoc properties (mostly path-related) + contact sequences + relational event models + ...
what we propose

deal with the stream directly

stream graphs and link streams

graph theory
network science

signal analysis, time series

wanted features: simple and intuitive, comprehensive, time-node consistent, generalizes graphs/signal
what we propose

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stream graphs and link streams

wanted features: simple and intuitive, comprehensive, time-node consistent, generalizes graphs/signal
graph-equivalent streams

stream with no dynamics:
  nodes always present,    ⇐⇒    graph
  either always or never linked

\begin{align*}
  \text{a} & \quad \cdots \quad \text{b} \\
  & \quad \cdots \quad \text{c} \\
  & \quad \cdots \quad \text{d} \\
  \text{e} & \quad \cdots
\end{align*}
graph-equivalent streams

**stream with no dynamics:**
- nodes always present,
- either always or never linked

\[ \iff \quad \text{graph} \]

\[
\begin{array}{cccccc}
& a & & & & \\
0 & 2 & 4 & 6 & 8 & \text{time} \\
\hline
b & a & & & & \\
c & a & a & & & \\
d & a & b & b & & \\
e & a & b & c & c & \\
\end{array}
\]

**stream properties** \( \iff \) **graph properties**

\( \iff \) \text{generalizes graph theory}
very careful generalization of the most basic concepts
stream graphs and link streams
numbers of nodes and links
clusters and induced sub-streams
density and paths

building blocks for higher-level concepts
neighborhood and degrees
clustering coefficient
betweenness centrality
many others

+ ensure consistency with graph theory
+ ensure classical relations are preserved
definition of stream graphs

Graph $G = (V, E)$ with $E \subseteq V \otimes V$

$uv \in E \iff u$ and $v$ are linked

Stream graph $S = (T, V, W, E)$

$T$: time interval, $V$: node set
$W \subseteq T \times V$, $E \subseteq T \times V \otimes V$

$(t, v) \in W \iff v$ is present at time $t$

$T_v = \{t, (t, v) \in W\}$

$(t, uv) \in E \iff u$ and $v$ are linked at time $t$

$T_{uv} = \{t, (t, uv) \in E\}$

$(t, uv) \in E$ requires $(t, u) \in W$ and $(t, v) \in W$
i.e. $T_{uv} \subseteq T_u \cap T_v$
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\[ T_{uv} = \{ t, (t, uv) \in E \} \]

\((t, uv) \in E \text{ requires } (t, u) \in W \text{ and } (t, v) \in W\)
i.e. \( T_{uv} \subseteq T_u \cap T_v \)
an example

\[ T = [0, 10] \quad V = \{a, b, c, d\} \]

\[ W = T \times \{a\} \cup ([0, 4] \cup [5, 10]) \times \{b\} \cup [4, 9] \times \{c\} \cup [1, 3] \times \{d\} \]

\[ T_a = T \quad T_b = [0, 4] \cup [5, 10] \quad T_c = [4, 9] \quad T_d = [1, 3] \]

\[ E = ([1, 3] \cup [7, 8]) \times \{ab\} \cup [6, 9] \times \{bc\} \cup [2, 3] \times \{bd\} \]

\[ T_{ab} = [1, 3] \cup [7, 8] \quad T_{bc} = [6, 9] \quad T_{bd} = [2, 3] \quad T_{ad} = \emptyset \]
a few remarks

works with... discrete time, continuous time, instantaneous interactions or with durations, directed, weighted, bipartite...

if \( \forall v, \ T_v = T \) then \( S \sim L = (T, V, E) \) is a link stream

if \( \forall u, v, \ T_{uv} \in \{ T, \emptyset \} \) then \( S \sim G = (V, E) \) is a graph-equivalent stream
size of a stream graph

How many nodes? How many links?

| $T_a$ | = 10 ≠ | $T_d$ | = 2

\[ n = |T_a| + |T_b| + |T_c| + |T_d| = 10 + 10 + 0.9 + 0.5 + 0.2 = 22.6 \text{ nodes} \]

\[ m = |T_{ab}| + |T_{bc}| + |T_{bd}| = 0.3 + 0.3 + 0.1 = 0.7 \text{ links} \]
Size of a stream graph

How many nodes? How many links?

\[ n = \sum_{v \in V} \frac{|T_v|}{|T|} \]

\[ n = \frac{|T_a|}{10} + \frac{|T_b|}{10} + \frac{|T_c|}{10} + \frac{|T_d|}{10} = 1 + 0.9 + 0.5 + 0.2 = 2.6 \text{ nodes} \]
size of a stream graph

How many nodes? How many links?

\[ n = \sum_{v \in V} \frac{|T_v|}{|T|} \]

\[ m = \sum_{uv \in V \otimes V} \frac{|T_{uv}|}{|T|} \]

\[ n = \frac{|T_a|}{10} + \frac{|T_b|}{10} + \frac{|T_c|}{10} + \frac{|T_d|}{10} = 1 + 0.9 + 0.5 + 0.2 = 2.6 \text{ nodes} \]

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clusters, sub-streams

Cluster in $G = (V, E)$: a subset of $V$.
Cluster in $S = (T, V, W, E)$: a subset of $W \subseteq T \times V$.

$$C = [0, 2] \times \{a\} \cup ([0, 2] \cup [6, 10]) \times \{b\} \cup [4, 8] \times \{c\}$$

$S(C)$ sub-stream induced by $C$

$S(C) = (T, V, C, E_C)$

$\leftrightarrow$ properties of (sub-streams induced by) clusters
clusters, sub-streams

Cluster in $G = (V, E)$: a subset of $V$.
Cluster in $S = (T, V, W, E)$: a subset of $W \subseteq T \times V$.

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$S(C)$ sub-stream induced by $C$
$$S(C) = (T, V, C, E_C)$$

→ properties of (sub-streams induced by) clusters
neighborhood

in $G = (V, E)$: $N(v) = \{u, uv \in E\}$

in $S = (T, V, W, E)$: $N(v) = \{(t, u), (t, uv) \in E\}$

$N(d) = ([2, 3] \cup [5, 10]) \times \{b\} \cup [5.5, 9] \times \{c\}$

$N(v)$ is a cluster
in G and in S:

\[ d(v) \text{ is the size of } N(v) \]

\[ N(d) = ([2, 3] \cup [5, 10]) \times \{b\} \cup [5.5, 9] \times \{c\} \]

\[ d(d) = \frac{|[2, 3] \cup [5, 10]|}{10} + \frac{|[5.5, 9]|}{10} = 0.6 + 0.35 = 0.95 \]

- degree distribution, average degree, etc
- if graph-equivalent stream then graph degree
- relation with \( n \) and \( m \)
in G:
proba two random nodes are linked
\[ \delta(G) = \frac{\text{nb links}}{\text{nb possible links}} = \frac{2 \cdot m}{n \cdot (n-1)} \]

in S:
proba two random nodes are linked at a random time instant
\[ \delta(S) = \frac{\text{nb links}}{\text{nb possible links}} = \frac{\sum_{uv \in V \otimes V} |T_{uv}|}{\sum_{uv \in V \otimes V} |T_u \cap T_v|} \]
Density

In $G$:
probabilty two random nodes are linked

$$\delta(G) = \frac{\text{nb links}}{\text{nb possible links}} = \frac{2 \cdot m}{n \cdot (n-1)}$$

In $S$:
probabilty two random nodes are linked at a random time instant

$$\delta(S) = \frac{\text{nb links}}{\text{nb possible links}} = \frac{\sum_{u \in V \otimes V} |T_{uv}|}{\sum_{u \in V \otimes V} |T_u \cap T_v|}$$

- if graph-equivalent stream then graph density
- relation with $n$, $m$, and average degree
in G: sub-graph of density 1
all nodes are linked together

in S: sub-stream of density 1
all nodes interact all the time
clustering coefficient

in G and in S:
density of the neighborhood

\( cc(v) = \delta(N(v)) \)

\[ N(d) = ([2, 3] \cup [5, 10]) \times \{b\} \cup [5.5, 9] \times \{c\} \]
clustering coefficient

in G and in S:

density of the neighborhood

\[ cc(v) = \delta(N(v)) \]

\[
N(d) = ([2, 3] \cup [5, 10]) \times \{b\} \cup [5.5, 9] \times \{c\}
\]

\[
cc(d) = \delta(N(d)) = \frac{|[6,9]|}{|[5.5,9]|} = \frac{6}{7}
\]
in $G$: 

from $a$ to $d$:  
$(a, b), (b, c), (c, d)$  
length: 3

→ shortest paths

in $S$: 

from $(1, d)$ to $(9, c)$:  
$(2, d, b), (3, b, a), (7.5, a, b), (8, b, c)$  
length: 4  
duration: 6  
→ shortest paths  
→ fastest paths
in $G$: from $a$ to $d$: $(a, b), (b, c), (c, d)$  
length: 3  
→ shortest paths

in $S$:  
from $(1, d)$ to $(9, c)$: $(2, d, b), (3, b, a), (7.5, a, b), (8, b, c)$  
length: 4  
duration: 6  
→ shortest paths  
→ fastest paths
betweenness centrality

in $G$:

\[ b(v) = \text{fraction of shortest paths from any } u \text{ to any } w \text{ in } V \text{ that involve } v \]

in $S$:

\[ b(t, v) = \text{fraction of shortest fastest paths from any } (i, u) \text{ to any } (j, w) \text{ in } W \text{ that involve } (t, v) \]
betweenness centrality

in $G$:

$$b(v) = \text{fraction of shortest paths from any } u \text{ to any } w \text{ in } V \text{ that involve } v$$

in $S$:

$$b(t, v) = \text{fraction of shortest fastest paths from any } (i, u) \text{ to any } (j, w) \text{ in } W \text{ that involve } (t, v)$$
many other concepts
relations vs interactions

graph/networks = relations
(like friendship)

dynamic graphs/networks = evolution of relations
(like new friends)

stream graphs / link streams = interactions
(like face-to-face contacts)

interactions = traces realiztion of relations?
link streams = traces of graphs/networks?

relations = consequences of interactions?
graphs/networks = traces of link streams?
we provide a language (set of concepts) that:

- makes it easy to deal with interaction traces,
- combines structure and dynamics in a consistent way,
- generalizes graphs / networks; signals / time series?
- meets classical and new algorithmic challenges,
- opens new perspectives for data analysis,
- clarifies the interplay interactions \( \leftrightarrow \) relations.

**studies in progress:** internet traffic, financial transactions, mobility/contacts, mailing-lists, sales, etc.
calls for papers

special issues of international journals

Theoretical Computer Science (TCS)

Link Streams: models and algorithms

Computer Networks

Link Streams: methods and case studies

deadline: July 1st

http://link-streams.com